Math 53: Multivariable Calculus **Sections 102, 103**

Worksheet for 2020-01-22

Conceptual questions

Question 1. Draw the parametric curve $x = e^t, y = e^{2t}$. (When not stated, you should assume that the t interval is the largest possible, in this case from $-\infty$ to ∞.)

Question 2. If $x = f(t), y = g(t)$ is some parametric curve, how does $x = f(3t), y = g(3t)$ compare?

Question 3. If $x = f(t), y = g(t)$ parametrizes the shape S, how could you parametrize the shape obtained by:

- first shifting by 2 units in the positive y direction,
- then stretching by a factor of 3 in the x direction,
- and lastly reflecting across the line $y = x$?

Question 4. If the equation $x^2 + y^2 = 5$ describes the shape S, what Cartesian equation describes the shape obtained by performing the steps in the previous question?

Question 5. Suppose a parametrization $x = f(t), y = g(t), a \le t \le b$ traces out a circle exactly once, ending where it started. One of the expressions $\int_a^b f(t)g'(t)\,dt$ and $\int_a^b g(t)f'(t)\,dt$ computes the area enclosed by the circle, and the other is its negative. Figure out which is which (it will depend on the direction that the circle is traced!).

Computations

Problem 1. Find a Cartesian equation for the parametric curve $x = t^3 + t, y = t^2 + 2$. Hint: compute x^2.

Problem 2. Find a parametrization for the curve $y^2 = x^3$.

Problem 3 (Stewart §10.2.54)**.** Compute the arclength of the "astroid" $x = \cos^3\theta, y = \sin^3\theta$ depicted in Figure 1. (Stewart §10.2.34 asks you for the area.)

Problem 4. There are two points on the curve

$$x = 2t^2, y = t - t^2, -\infty < t < \infty$$

where the tangent line passes through the point $(10, -2)$. Find these two points.

Problem 5. Finding parametrizations given Cartesian equations is very hard in general (if it is even possible). Here are two tricky ones:

- Find a parametrization for the entire curve $y^2 = x^2(x + 1)$.
- Parametrize some part of the circle $x^2 + y^2 = 1$ using only rational functions—i.e. quotients of polynomials in t. In particular, do not use square roots or trigonometric functions.

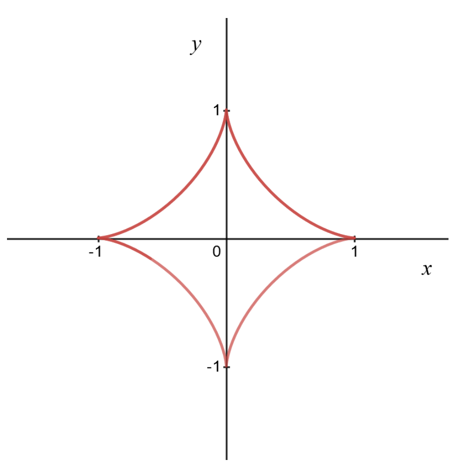

FIGURE 1. Problem 3

Below are the numerical answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1. Right half of the parabola $y = x^2$, not including the origin.

Question 2. Same curve traced out three times as fast.

Question 3. $x = g(t) + 2, y = 3f(t)$.

Question 4. $\left(\frac{y}{3}\right)^2 + (x-2)^2 = 5$.

Question 5. If the circle is traced clockwise, then $\int_a^b g(t)f'(t)\,\mathrm{d}t$ gives the area. If the circle is traced counterclockwise, then $\int_a^b f(t)g'(t)\,\mathrm{d}t$ gives the area.

Answers to computations

Problem 1. $x^2 = (y-2)^3 + 2(y-2)^2 + (y-2)$.

Problem 2. Many possible answers; probably simplest are $x = t^{2/3}, y = t$ or $x = t^2, y = t^3$. (Note that $x = t, y = t^{3/2}$ would only make sense for $t \geq 0$, and would only trace out half of the curve!)

Problem 3. 6

Problem 4. $(2, 0)$ and $(50, -20)$.

Problem 5. Many possible answers; here are the “standard” ones.

- $x = t^2 - 1, y = t(t^2 - 1)$.
- $x = \frac{2t}{t^2+1}, y = \frac{t^2-1}{t^2+1}$.